2.7: Euler's method

Given $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$.

How can we estimate the path of a solution, if we can't solve?

Ideas?

Observe:

1. We have a point on the solutions, $y(t_0) = y_0$.
2. And we know the *slope*, $\frac{dy}{dt}$, at any point.

Thoughts?

Recall:

The equation of a line looks like

$$y = y_0 + m(t - t_0).$$

For a tangent line, the slope is

$$m = \frac{dy}{dt} = f(t_0, y_0)$$

which we can compute without solving the differential equation!

So we can get the tangent line and take a “small step” in the direction of the tangent line. Then repeat.

(Why a small step?).

Summary (Euler's Method)

To numerically estimate the sol'n:

1. Choosing a step size, *h*.
2. Compute slope
$$\frac{dy}{dt} = f(t_0, y_0).$$
3. Use tangent line approx.:
$$y_1 = y_0 + f(t_0, y_0)h.$$
4. Repeat steps 2-3 to get y_2, and so on.

Briefly,

$$y_{n+1} = y_n + f(t_n, y_n)h$$
$$t_{n+1} = t_n + h$$

Example:

Given $\frac{dy}{dt} = 2t - y$, $y(2) = 4$.

Estimate $y(4)$ using Euler's method with step size h = 0.5.

For comparison (obtained by integrating factors)

Actual solution:

$y(t) = 2(t + e^{2-t} - 1)$,

so y(4) = 6.270671

t_n	y_n	$f(t_n, y_n) = slope$	$f(t_n, y_n)h = \Delta y$
2	4	2(2)-(4) = 0	(0)(0.5) = 0
2.5	4	2(2.5)-(4) = 1	(1)(0.5) = 0.5
3	4.5	2(3.0)-(4.5) = 1.5	(1.5)(0.5) = 0.75
3.5	5.25	2(3.5)-(5.25) = 1.75	(1.75)(0.5) = 0.875
4	???		

Conclusion: y(4) ≈